POINCARÉ-TYPE INEQUALITIES FOR SINGULAR STABLE-LIKE DIRICHLET FORMS

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Abstract. This paper is concerned with a class of singular stable-like Dirichlet forms on $\mathbb{R}^d$, which are generated by d independent copies of a one-dimensional symmetric α-stable process, and whose Lévy jump kernel measure is concentrated on the union of the coordinate axes. Explicit and sharp criteria for Poincaré inequality, super Poincaré inequality and weak Poincaré inequality of such singular Dirichlet forms are presented. When the reference measure is a product measure on $\mathbb{R}^d$, we also consider the entropy inequality for the associated Dirichlet forms, which is similar to the log-Sobolev inequality for local Dirichlet forms, and enjoys the tensorisation property.

Keywords: singular stable-like non-local Dirichlet form; (super/weak) Poincaré inequality; entropy inequality; tensorisation property; Lyapunov type conditions

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1. Introduction and Main Results

1.1. **Background for Functional Inequalities of Singular Stable-like Dirichlet Forms.** Let $\mu_V(dx) = e^{-V(x)}\,dx$ be a probability measure on $\mathbb{R}^d$, where $V \in C^1(\mathbb{R}^d)$. In the past few years functional inequalities for the following local Dirichlet form $(D_B, \mathscr{D}(D_B))$:

$$D_B(f,f) = \int |\nabla f(x)|^2\,\mu_V(dx),$$
$$\mathscr{D}(D_B) = \big\{ f \in L^2(\mathbb{R}^d;\mu) : D_B(f,f) < \infty \big\}$$

have been intensely investigated by several probabilists. One of the motivations comes from the study of the ergodicity for diffusion semigroups associated with the second order elliptic operator

$$L_B f = \Delta f - \nabla V \cdot \nabla f,$$

which is the generator of the Dirichlet form $(D_B, \mathscr{D}(D_B))$, and also is the generator of the following stochastic differential equation

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)\,dt,$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}^d$. Note that the coordinate processes of $(B_t)_{t\geqslant 0}$ are d independent copies of a one-dimensional Brownian motion. One dimensional Brownian motion is a member of the class of one dimensional strong

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Markov processes, called symmetric α-stable processes on $\mathbb{R}$. A one dimensional symmetric α-stable process with index $\alpha \in (0,2]$ is the Lévy process $(Y_t)_{t\geqslant 0}$ so that

$$\mathbb{E}\Big[e^{i\xi(Y_t-Y_0)}\Big] = e^{-t|\xi|^\alpha} \quad \text{for } t>0 \text{ and } \xi\in\mathbb{R}.$$

When $\alpha = 2$, $(Y_t)_{t\geqslant 0}$ is just a Brownian motion but running at twice the speed. However for $\alpha \in (0,2)$, $(Y_t)_{t\geqslant 0}$ is a purely discontinuous Lévy process with no drift, no Gaussian part, and Lévy measure

$$n(dh) = c_\alpha/|h|^{1+\alpha}\, dh,$$

where $c_\alpha = \frac{\alpha 2^{\alpha-1}\Gamma((1+\alpha)/2)}{\pi^{1/2}\Gamma(1-\alpha/2)}$. Recently there has been intense interest on the study of processes with jumps. So it is natural to ask functional inequalities as above when Brownian motion $(B_t)_{t\geqslant 0}$ is replaced by d independent copies of a one-dimensional symmetric α-stable process. This is the topic of this paper.

We now give a more precise motivation of this paper. For any $t>0$, let $Z_t = (Z_t^1, \cdots, Z_t^d)$ be a vector of d independent one-dimensional symmetric α-stable processes with index $\alpha\in(0,2)$. Consider the following stochastic differential equation (SDE)

$$dX_t = dZ_t + b(X_t)\, dt \tag{1.1}$$

with some regularity drift term b, such that the equation (1.1) has a unique weak (or strong) solution $(X_t)_{t\geqslant 0}$ and a unique invariant probability measure μ. (Surely there is a close relation between the drift term b and the invariant measure μ, and we will discuss it in another paper). The existence of unique weak and strong solution to the SDE (1.1) have been studied in [1] and [13], respectively. Now, let $(P_t)_{t\geqslant 0}$ be the semigroup of the process $(X_t)_{t\geqslant 0}$ on $L^2(\mathbb{R}^d;\mu)$. Then, for any $f\in C_c^\infty(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P_s f(x)\, ds = \mu(f).$$

We are interested in measuring the way of P_t converging to its equilibrium distribution μ.

For simplicity, below we take the $L^2(\mathbb{R}^d;\mu)$-norm for example, i.e. to consider the bound for $\|P_t(f)-\mu(f)\|_{L^2(\mathbb{R}^d;\mu)}$ for $f\in L^2(\mathbb{R}^d;\mu)$. Note that the process $(Z_t)_{t\geqslant 0}$ is a d-dimensional Lévy process, and it picks a coordinate at random from $\{1,\dots,d\}$ and then jumps a positive or negative distance in that direction. Therefore, the Lévy measure for $(Z_t)_{t\geqslant 0}$ is more singular than that of the spherically symmetric α-stable process (see $\nu_S(dz)$ below), and it is concentrated on the union of the coordinate axes, a one-dimensional subset of $\mathbb{R}^d$; that is, the density with respect to the Lebesgue measure of the Lévy measure for the process $(Z_t)_{t\geqslant 0}$ is given by

$$c_\alpha\left(\delta_{\{x_2=0,\dots,x_d=0\}}\frac{1}{|x_1|^{1+\alpha}} + \dots + \delta_{\{x_1=0,\dots,x_{d-1}=0\}}\frac{1}{|x_d|^{1+\alpha}}\right),$$

where δ_A is Dirac measure of the set $A\subset\mathbb{R}^d$. Hence, the generator of the process $(X_t)_{t\geqslant 0}$ enjoys the following expression

$$Lf(x) = \langle b(x), \nabla f(x)\rangle + \sum_{i=1}^d \int_{\mathbb{R}}\Big(f(x+ze_i) - f(x) - z\mathbb{1}_{\{|z|\leqslant 1\}}\nabla f(x)\cdot e_i\Big)\frac{c_\alpha\, dz}{|z|^{1+\alpha}},$$

where $e_i = (\overbrace{0,\dots,0}^{i-1},1,\overbrace{0,\dots,0}^{d-i})$ for $1 \leqslant i \leqslant d$. Furthermore, by some formal calculation, for every $f \in C_c^\infty(\mathbb{R}^d)$ such that $\mu(f)=0$,

$$\begin{aligned}
&\frac{d}{dt}\mu((P_tf)^2)\\
&=2\int (LP_tf)P_tf\,d\mu\\
&=-c_\alpha\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{\big(P_tf(x+ze_i)-P_tf(x)\big)^2}{|z|^{1+\alpha}}\,dz\,\mu(dx)+\int L((P_tf)^2)\,d\mu\\
&=-c_\alpha\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{\big(P_tf(x+ze_i)-P_tf(x)\big)^2}{|z|^{1+\alpha}}\,dz\,\mu(dx),
\end{aligned}$$

where in the second equality we have used the fact that

$$L(f^2)=2fLf+c_\alpha\sum_{i=1}^d\int_{\mathbb{R}}\frac{\big(f(x+ze_i)-f(x)\big)^2}{|z|^{1+\alpha}}\,dz,$$

and the third equality is due to that μ is an invariant probability measure and

$$\int L((P_tf)^2)\,d\mu=0.$$

Therefore, if one can prove

$$\mu(f^2)\leqslant\frac{Cc_\alpha}{2}\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{\big(f(x+ze_i)-f(x)\big)^2}{|z|^{1+\alpha}}\,dz\,\mu(dx),\quad \mu(f)=0 \tag{1.2}$$

holds for some constant $C>0$, then for every $f\in C_c^\infty(\mathbb{R}^d)$ such that $\mu(f)=0$,

$$\frac{d}{dt}\mu((P_tf)^2)\leqslant-\frac{2}{C}\mu((P_tf)^2).$$

This implies that for every $f\in C_c^\infty(\mathbb{R}^d)$ such that $\mu(f)=0$,

$$\mu((P_tf)^2)\leqslant e^{-2t/C}\mu(f^2),\quad t>0.$$

In particular, we have

$$\|P_tf-\mu(f)\|_{L^2(\mathbb{R}^d;\mu)}\leqslant e^{-t/C}\|f-\mu(f)\|_{L^2(\mathbb{R}^d;\mu)},\quad t>0, f\in L^2(\mathbb{R}^d;\mu),$$

which is our desired assertion. In particular, (1.2) motivates us to study the following bilinear form

$$D(f,f)=\frac{1}{2}\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu(dx).$$

Let $\mathscr{D}(D)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ under the D_1-norm

$$\|f\|_{D_1}:=(\|f\|^2_{L^2(\mathbb{R}^d;\mu)}+D(f,f))^{1/2}.$$

Then, according to [7, Example 1.2.4], $(D,\mathscr{D}(D))$ is regular symmetric Dirichlet form on $L^2(\mathbb{R}^d;\mu)$. In this setting, (1.2) is the Poincaré inequality for the Dirichlet form $(D,\mathscr{D}(D))$, which will be stated explicitly in Theorem 1.1 below.

The main goal of this paper is to prove various Poincaré type inequalities for the Dirichlet form $(D, \mathscr{D}(D))$. Recently, explicit criteria have been presented in [4, 6, 17] for functional inequalities of the following (standard) stable-like Dirichlet form

$$D_S(f,f) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f(x+z)-f(x))^2}{|z|^{d+\alpha}}\,dz\,\mu(dx), \quad f\in\mathscr{D}(D_S),$$

where $\mathscr{D}(D_S)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ under the $D_{S,1}$-norm

$$\|f\|_{D_{S,1}} := (\|f\|^2_{L^2(\mathbb{R}^d;\mu)} + D_S(f,f))^{1/2}.$$

Different from the singular stable-like Dirichlet form $(D, \mathscr{D}(D))$ considered in the present paper, the Lévy jump kernel of $(D_S, \mathscr{D}(D_S))$ is associated with spherically symmetric α-stable processes in [17], and it is given by

$$\nu_S(dz) = \frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,dz,$$

where $C_{d,\alpha} = \frac{\alpha 2^{\alpha-1}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}$.

Comparing with the method for the proofs of Poincaré type inequalities for $(D_S, \mathscr{D}(D_S))$ in [17], in order to get the corresponding functional inequalities for $(D, \mathscr{D}(D))$, we will face with two fundamental differences:

(1) The efficient approach to yield functional inequalities for $(D_S, \mathscr{D}(D_S))$ is to check the Lyapunov type condition for the associated generator, which heavily depends on the corresponding Lévy jump kernel $\nu_S(dz)$. In particular, the Lyapunov function ϕ we choose in [17] is of the form $\phi(x) = |x|^\beta$ with some constant $\beta \in (0, 1\wedge\alpha)$ for $|x|$ large enough. Such test function ϕ is useful for the generator of $(D, \mathscr{D}(D))$, but the argument of [17, Proposition 2.3] does not work in the present setting.

(2) Another ingredient to obtain Poincaré inequality and super Poincaré inequality for $(D_S, \mathscr{D}(D_S))$ is to prove the local Poincaré inequality and the local super Poincaré inequality. The local super Poincaré inequality for $(D_S, \mathscr{D}(D_S))$ is derived from the classical Sobolev inequality for fractional Laplacians; while the local Poincaré inequality for $(D_S, \mathscr{D}(D_S))$ is easily obtained by applying the Cauchy-Schwarz inequality to local variance. However we are unable to use these approaches here, since the Lévy jump kernel for the Dirichlet form $(D, \mathscr{D}(D))$ is much more singular.

Due to the above differences and difficulties, obtaining the criteria for Poincaré inequality and super Poincaré inequality for $(D, \mathscr{D}(D))$ requires new approaches and ideas, which include the following two points:

(3) The new choice of Lyapunov function ϕ (a little different from that in [17]) for the generator associated with $(D, \mathscr{D}(D))$. Some more refined calculations are required, due to the character of Lévy jump kernel for the Dirichlet form $(D, \mathscr{D}(D))$. (See Proposition 2.3 and its proof.)

(4) The local super Poincaré inequality for $(D, \mathscr{D}(D))$, where a new direct proof of the local functional inequality for singular non-local Dirichlet forms is given. (See Proposition 2.4.) An application of the more recent result on the equivalence of defective Poincaré inequality and true Poincaré inequality for symmetric conservative Dirichlet forms developed in [15], yields the desired Poincaré inequality for $(D, \mathscr{D}(D))$, and circumvents the difficulty of proving

the local Poincaré inequality for $(D, \mathscr{D}(D))$. (See the proof of Theorem 1.1(1).)

1.2. **Main Results: Criteria for Functional Inequalities of Singular Stable-like Dirichlet Forms.** Throughout the paper, we always suppose that $\mu(dx) = \mu_V(dx) := e^{-V}\,dx$ *is a probability measure on* $\mathbb{R}^d$, *such that* e^{-V} *is a bounded measurable function on* $\mathbb{R}^d$. We emphasize that unlike [17] no C^1-regularity on V is needed in the present paper. For any $x \in \mathbb{R}^d$, set

$$\Gamma^V_{\inf}(x) := \inf_{1\leqslant i\leqslant d:\, |x_i|\geqslant |x|/\sqrt{d}} \ \inf_{|u_i|\leqslant 1} e^{-V(x_1,\cdots,x_{i-1},u_i,x_{i+1},\cdots,x_d)},$$

$$\Gamma^V_{\sup}(x) := \sup_{1\leqslant i\leqslant d} \ \sup_{|u_i|\geqslant |x_i|} e^{-V(x_1,\cdots,x_{i-1},u_i,x_{i+1},\cdots,x_d)}$$

and

$$\Lambda(x) := \frac{e^{V(x)}\Gamma^V_{\inf}(x)}{(1+|x|)^{1+\alpha}}.$$

Furthermore, we define

$$\Phi(r) = \inf_{|x|\geqslant r} \Lambda(x), \quad r > 0.$$

We are now in a position to state the main result in our paper.

Theorem 1.1. *Suppose that there exists a constant* $\gamma \in (0, \alpha \wedge 1)$ *such that*

$$\limsup_{|x|\to\infty} \frac{|x|^{1+\alpha-\gamma}\Gamma^V_{\sup}(x)}{\Gamma^V_{\inf}(x)} = 0. \tag{1.3}$$

We have the following statements.

(1) *If* $\lim_{r\to\infty}\Phi(r) > 0$, *then the following Poincaré inequality*

$$\mu_V(f^2) \leqslant CD(f,f) + \mu_V(f)^2, \quad f \in \mathscr{D}(D) \tag{1.4}$$

holds with some constant $C > 0$.

(2) *If* $\lim_{r\to\infty}\Phi(r) = \infty$, *then the following super Poincaré inequality*

$$\mu_V(f^2) \leqslant sD(f,f) + \beta(s)\mu_V(|f|)^2, \quad s > 0, f \in \mathscr{D}(D) \tag{1.5}$$

holds with

$$\beta(s) = C_1(1 + s^{-d/\alpha})\frac{\left(\sup_{|x|\leqslant 2\sqrt{d}\Phi^{-1}(C_2(1+1/s))} e^{V(x)}\right)^{2+d/\alpha}}{\left(\inf_{|x|\leqslant \Phi^{-1}(C_2(1+1/s))} e^{V(x)}\right)^{1+d/\alpha}}$$

for some constants C_1 *and* $C_2 > 0$, *where* Φ^{-1} *is the generalized inverse of* Φ, *i.e.* $\Phi^{-1}(r) = \inf\{s \geqslant 0 : \Phi(s) \geqslant r\}$.

The following corollary shows that Theorem 1.1 is sharp in some situation.

Corollary 1.2. *Let*

$$e^{-V(x)} = C_{\varepsilon_1,\ldots,\varepsilon_d}\prod_{i=1}^d (1+|x_i|)^{-(1+\varepsilon_i)}$$

with $\varepsilon_i > 0$ *for all* $1 \leqslant i \leqslant d$.

(1) *The Poincaré inequality* (1.4) *holds with some constant* $C > 0$ *if and only if* $\varepsilon_i \geqslant \alpha$ *for all* $1 \leqslant i \leqslant d$.

(2) *The super Poincaré inequality* (1.5) *holds with some function* $\beta:(0,\infty)\to(0,\infty)$ *if and only if* $\varepsilon_i>\alpha$ *for all* $1\leqslant i\leqslant d$*, and in this case there exists a constant* $c>0$ *such that the super Poincaré inequality* (1.5) *holds with*

$$\beta(r)\leqslant c\left(1+r^{-\left(\frac{d}{\alpha}+\frac{(2\alpha+d)\sum_{i=1}^d(1+\varepsilon_i)}{\alpha(\varepsilon_*-\alpha)}\right)}\right),\quad r>0,$$

where $\varepsilon_*=\min_{1\leqslant i\leqslant d}\varepsilon_i$*; and equivalently,*

$$\begin{aligned}\|P_t\|_{L^1(\mathbb{R}^d;\mu_V)\to L^\infty(\mathbb{R}^d;\mu_V)}:&=\sup_{f\in L^1(\mathbb{R}^d;\mu_V)}\|P_tf\|_{L^\infty(\mathbb{R}^d;\mu_V)}\\&\leqslant\lambda\left(1+t^{-\left(\frac{d}{\alpha}+\frac{(2\alpha+d)\sum_{i=1}^d(1+\varepsilon_i)}{\alpha(\varepsilon_*-\alpha)}\right)}\right),\quad t>0\end{aligned}$$

holds with some constant $\lambda>0$.

We give two remarks on Corollary 1.2, which point out the difference between Dirichlet form $(D,\mathscr{D}(D))$ and $(D_S,\mathscr{D}(D_S))$. Since $(D,\mathscr{D}(D))$ is just $(D_S,\mathscr{D}(D_S))$ when $d=1$, we assume that $d\geqslant 2$ below. Both reference measures μ in $(D,\mathscr{D}(D))$ and $(D_S,\mathscr{D}(D_S))$ are given by $\mu(dx)=\mu_V(dx)=e^{-V(x)}\,dx$.

(i) Let

$$e^{-V(x)}=C_{\varepsilon_1,\dots,\varepsilon_d}\prod_{i=1}^d(1+|x_i|)^{-(1+\varepsilon_i)}$$

with $\varepsilon_i\geqslant\alpha$ for all $1\leqslant i\leqslant d$. We know from Corollary 1.2 above that the Poincaré inequality holds for $(D,\mathscr{D}(D))$; however, we do not know whether the Poincaré inequality holds for the standard Dirichlet form $(D_S,\mathscr{D}(D_S))$, because the assumptions of [17, Theorem 1.1] are not satisfied.

(ii) Let

$$e^{-V(x)}=C_{\varepsilon,d}(1+|x|)^{-(d+\varepsilon)}$$

with $\varepsilon>0$. Then, according to [17, Corollary 1.2], we know that the Poincaré inequality holds for the Dirichlet form $(D_S,\mathscr{D}(D_S))$ if and only if $\varepsilon\geqslant\alpha$, and the super Poincaré inequality holds for $(D_S,\mathscr{D}(D_S))$ if and only if $\varepsilon>\alpha$. On the other hand, we do not know whether the Poincaré inequality holds for the Dirichlet form $(D,\mathscr{D}(D))$ for any $\varepsilon>0$, since the assumptions of Theorem 1.1 do not hold for all $\varepsilon>0$.

The following corollary further indicates that the conclusions of Theorem 1.1 are explicit in some setting.

Corollary 1.3. *Let*

$$e^{-V(x)}=C_{\varepsilon_1,\dots,\varepsilon_d,\alpha}\prod_{i=1}^d(1+|x_i|)^{-(1+\alpha)}\log^{-\varepsilon_i}(e+|x_i|)$$

with $\varepsilon_i\in\mathbb{R}$ *for all* $1\leqslant i\leqslant d$.

(1) *The Poincaré inequality* (1.4) *holds for some constant* $C>0$ *if and only if* $\varepsilon_i\geqslant 0$ *for all* $1\leqslant i\leqslant d$.

(2) *The super Poincaré inequality* (1.5) *holds for some function* $\beta:(0,\infty)\to(0,\infty)$ *if and only if* $\varepsilon_i>0$ *for all* $1\leqslant i\leqslant d$*, and in this case there exists a constant* $c>0$ *such that the super Poincaré inequality* (1.5) *holds with*

$$\beta(r)\leqslant\exp\Big(c\big(1+r^{-1/\varepsilon_*}\big)\Big),\quad r>0$$

for some constants $c > 0$ and $\varepsilon_* = \min_{1\leqslant i\leqslant d} \varepsilon_i$, so that when $\varepsilon_* > 1$,

$$\|P_t\|_{L^1(\mathbb{R}^d;\mu_V)\to L^\infty(\mathbb{R}^d;\mu_V)} \leqslant \exp\left(\lambda(1 + t^{-1/(\varepsilon_*-1)})\right), \quad t > 0$$

holds for some constant $\lambda > 0$. The rate function β above is sharp in the sense that (1.5) does not hold if

$$\lim_{r\to 0} r^{1/\varepsilon_*} \log \beta(r) = 0.$$

In particular, the following log-Sobolev inequality

$$\mu_V(f^2 \log f^2) \leqslant CD(f, f), \quad f \in \mathscr{D}(D), \mu_V(f^2) = 1$$

holds for some constant $C > 0$ if and only if $\varepsilon_i \geqslant 1$ for all $1 \leqslant i \leqslant d$.

Both Corollaries 1.2 and 1.3 are concerned with Poincaré-type inequalities for product measures. As mentioned in the remark after the proof of Theorem 3.1 in Section 3, Poincaré inequalities for Corollaries 1.2(1) and 1.3(1) can be obtained from the results of [17] in one-dimensional setting and the well-known tensorisation procedure. In the following example, we consider product measure with variable order. We mention that the Poincaré inequality for such measure can not be deduced directly by tensorisation argument.

Example 1.4. Let

$$e^{-V(x)} = C \prod_{i=1}^{d} (1 + |x_i|)^{-(1+a_i(x))},$$

where for all $1 \leqslant i \leqslant d$, a_i is a bounded Borel measurable function such that $\inf_{x\in\mathbb{R}^d} a_i(x) > 0$, and $C > 0$ is the normalizing constant. Define

$$a_j^*(x|u_i) := a_j((x_1, \dots, x_{i-1}, u_i, x_{i+1}, \dots x_d)), \quad x \in \mathbb{R}^d, u_i \in \mathbb{R}, 1 \leqslant i, j \leqslant d.$$

Suppose that for $|x|$ large enough and for all $1 \leqslant j \leqslant d$,

$$M_j(x) := \inf_{1\leqslant i\leqslant d} \inf_{|u_i|\geqslant|x_i|} a_j^*(x|u_i) \geqslant \sup_{1\leqslant i\leqslant d:\, |x_i|\geqslant|x|/\sqrt{d}} \sup_{|u_i|\leqslant 1} a_j^*(x|u_i) =: N_j(x). \tag{1.6}$$

Then, we have the following statements.

(1) If for $|x|$ large enough,

$$A(x) := \inf_{1\leqslant i\leqslant d:|x_i|\geqslant|x|/\sqrt{d}} \sup_{|u_i|\leqslant 1} a_i^*(x|u_i) \geqslant \alpha, \tag{1.7}$$

then the Poincaré inequality (1.4) holds with some constant $C > 0$.

(2) If

$$A^* := \liminf_{|x|\to\infty} A(x) > \alpha,$$

then for any $\varepsilon > 0$, there is a constant $c = c(\varepsilon) > 0$ such that the super Poincaré inequality (1.5) holds with

$$\beta(r) \leqslant c\left(1 + r^{-\left(\frac{d}{\alpha} + \frac{(2\alpha+d)\sum_{i=1}^d (1+B_i)}{\alpha(A^*-\varepsilon-\alpha)}\right)}\right), \quad r > 0, \tag{1.8}$$

where $B_i = \sup_{x\in\mathbb{R}^d} a_i(x)$ for all $1 \leqslant i \leqslant d$. If moreover $\inf_{|x|\geqslant r} A(x)$ is constant for r large enough, then (1.8) is satisfied with $\varepsilon = 0$, i.e.

$$\beta(r) \leqslant c_0\left(1 + r^{-\left(\frac{d}{\alpha} + \frac{(2\alpha+d)\sum_{i=1}^d (1+B_i)}{\alpha(A^*-\alpha)}\right)}\right), \quad r > 0.$$

The remainder of this paper is arranged as follows. The next section is devoted to the preliminary analysis on singular stable-like Dirichlet form $(D, \mathscr{D}(D))$. The Lyapunov type drift condition for the associated truncated Dirichlet form $(D_{>1}, \mathscr{D}(D_{>1}))$ is established, and the local super Poincaré inequality for $(D, \mathscr{D}(D))$ is also presented. In Section 3, we will prove Theorem 1.1 and Corollaries 1.2, 1.3 and Example 1.4. In particular, on the one hand, when the reference measure is a product measure on $\mathbb{R}^d$, the entropy inequality for $(D, \mathscr{D}(D))$ is considered here, which shows that $(D, \mathscr{D}(D))$ enjoys the tensorisation property; on the other hand, the weak Poincaré inequality for $(D, \mathscr{D}(D))$ is also included, which can be regarded as a complement of Theorem 1.1.

2. Preliminary Analysis on Singular Stable-like Dirichlet Forms

Let $\mu_V(dx) = e^{-V(x)}\,dx$ be a probability measure on $\mathbb{R}^d$ such that e^{-V} is a bounded measurable function. For any $f \in C_b^1(\mathbb{R}^d)$, since

$$\begin{aligned}\sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}} &\frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx)\\ &\leqslant 4(\|f\|_\infty \vee \|\nabla f\|_\infty)^2 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}} \frac{1\wedge z^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx)\\ &< \infty,\end{aligned}$$

we can well define

$$D(f,f) := \frac{1}{2}\sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}} \frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx).$$

For any x, $y \in \mathbb{R}^d$, set

$$\begin{aligned}J(x,y) &:= \frac{1}{2}\big(e^{V(x)}+e^{V(y)}\big)\\ \times &\left(\delta_{\{x_2-y_2=0,\dots,x_d-y_d=0\}}\frac{1}{|x_1-y_1|^{1+\alpha}} + \dots + \delta_{\{x_1-y_1=0,\dots,x_{d-1}-y_{d-1}=0\}}\frac{1}{|x_d-y_d|^{1+\alpha}}\right),\end{aligned}$$

where δ_A is Dirac measure of the set A. Then,

$$D(f,f) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (f(x)-f(y))^2 J(x,y)\,\mu_V(dx)\,\mu_V(dy),$$

and $J(x,y)$ is the associated Lévy jump kernel measure. Furthermore, it is easy to check that

$$x \mapsto \int \big(1\wedge|x-y|^2\big)J(x,y)\,\mu_V(dy) \in L^1(\mathbb{R}^d;\mu_V).$$

Let $\mathscr{D}(D)$ be the closure of $C_c^\infty(\mathbb{R}^d)$ under the D_1-norm

$$\|f\|_{D_1} := \sqrt{\|f\|^2_{L^2(\mathbb{R}^d;\mu_V)} + D(f,f)}.$$

Then, we know from [7, Example 1.2.4] that $(D, \mathscr{D}(D))$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d;\mu_V)$. Furthermore, denote by $(P_t)_{t\geqslant 0}$ the semigroup on $L^2(\mathbb{R}^d;\mu_V)$ associated with $(D, \mathscr{D}(D))$, which can be extended into $L^\infty(\mathbb{R}^d;\mu_V)$, e.g. see [7, Page 56]. Since μ_V is a symmetric and invariant probability measure of $(P_t)_{t\geqslant 0}$, $1 = \mu_V(1) = \mu_V(P_t 1)$ for each $t>0$, which implies that $P_t 1(x) = 1$ for all $t > 0$ and almost all $x \in \mathbb{R}^d$. Then, the Dirichlet form $(D, \mathscr{D}(D))$ is conservative.

To deal with functional inequalities for the Dirichlet form $(D, \mathscr{D}(D))$, we will make full use of the truncation approach. For this, we define for any $f \in C_b^1(\mathbb{R}^d)$,

$$D_{>1}(f,f) := \frac{1}{2}\sum_{i=1}^d \int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}} \frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx).$$

Let $\mathscr{D}(D_{>1})$ be the closure of $C_c^\infty(\mathbb{R}^d)$ under the norm

$$\|f\|_{D_{>1},1} := \sqrt{\|f\|^2_{L^2(\mathbb{R}^d;\mu_V)} + D_{>1}(f,f)}.$$

It is clear that $D_{>1}(f,f) \leqslant D(f,f)$, and so $\mathscr{D}(D) \subset \mathscr{D}(D_{>1})$. From this, we also can easily conclude that $(D_{>1}, \mathscr{D}(D_{>1}))$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d;\mu_V)$.

Denote by $B(\mathbb{R}^d)$ the set of measurable functions on $\mathbb{R}^d$, and by $B_b(\mathbb{R}^d)$ the set of bounded measurable functions on $\mathbb{R}^d$. For any $f \in B_b(\mathbb{R}^d)$, define

$$L_{>1}f(x) := \frac{1}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \big(f(x+ze_i)-f(x)\big)\frac{e^{V(x)-V(x+ze_i)}+1}{|z|^{1+\alpha}}\,dz.$$

We have

Proposition 2.1. (1) *For any f, $g \in B_b(\mathbb{R}^d)$,*

$$D_{>1}(f,g) = -\int f L_{>1} g\, d\mu_V.$$

(2) *For $0<\gamma<\alpha$, let*

$$\mathscr{C}_\gamma := \Big\{ g \in B(\mathbb{R}^d) : \text{ there exists a constant } C>0 \text{ such that } \\ |g(x)-g(y)| \leqslant C|x-y|^\gamma \text{ for any } x,y\in\mathbb{R}^d \text{ with } |x-y|>1 \Big\}.$$

Then, $B_b(\mathbb{R}^d) \subset \mathscr{C}_\gamma$, and for any $g \in \mathscr{C}_\gamma$, $L_{>1}g$ exists pointwise as a locally bounded function. Moreover for any $f \in B_b(\mathbb{R}^d)$ and any $g \in \mathscr{C}_\gamma$,

$$\sum_{i=1}^d \int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}} \frac{|f(x+ze_i)-f(x)||g(x+ze_i)-g(x)|}{|z|^{1+\alpha}}\,dz\,\mu_V(dx) < \infty$$

and

$$\begin{aligned} -&\int f(x)L_{>1}g(x)\,\mu_V(dx) \\ &= \frac{1}{2}\sum_{i=1}^d \int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}} \frac{(f(x+ze_i)-f(x))(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}\,dz\,\mu_V(dx). \end{aligned}$$

Proof. (1) We first note that for any $g \in B_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} |L_{>1}g(x)| &\leqslant \frac{1+\|e^{-V}\|_\infty e^{V(x)}}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \frac{|g(x+ze_i)-g(x)|}{|z|^{1+\alpha}}\,dz \\ &\leqslant \frac{2d}{\alpha}\big(1+\|e^{-V}\|_\infty e^{V(x)}\big)\|g\|_\infty, \end{aligned}$$

which implies that $L_{>1}g$ is well defined. On the other hand, for any $f,\, g\in B_b(\mathbb{R}^d)$,

$$\begin{aligned}\Big|\int fL_{>1}g\,d\mu_V\Big| \leqslant&\|f\|_\infty\|g\|_\infty\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{1}{|z|^{1+\alpha}}(e^{-V(x)}+e^{-V(x+ze_i)})\,dz\,dx\\ =&\|f\|_\infty\|g\|_\infty\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{1}{|z|^{1+\alpha}}e^{-V(x)}\,dz\,dx\\ &+\|f\|_\infty\|g\|_\infty\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{1}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\,dx\\ =&2\|f\|_\infty\|g\|_\infty\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{e^{-V(x)}}{|z|^{1+\alpha}}\,dz\,dx\\ =&\frac{4d\|f\|_\infty\|g\|_\infty}{\alpha},\end{aligned}$$

where the second equality follows from the change of the variable $x\mapsto x+ze_i$ in the second term of the first equality. Thus, $\int fL_{>1}g\,d\mu_V$ is also well defined.

Furthermore, for any $f,\, g\in B_b(\mathbb{R}^d)$,

$$\begin{aligned}&-\int f(x)L_{>1}g(x)\,\mu_V(dx)\\ &\quad=-\frac12\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{f(x)(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}\Big(e^{-V(x)}+e^{-V(x+ze_i)}\Big)\,dz\,dx.\end{aligned}$$

Changing the variables $z\to -z$ and $x\to x+ze_i$ in the right hand side, we get

$$\begin{aligned}&-\int f(x)L_{>1}g(x)\,\mu_V(dx)\\ &=-\frac12\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{f(x+ze_i)(g(x)-g(x+ze_i))}{|z|^{1+\alpha}}\Big(e^{-V(x)}+e^{-V(x+ze_i)}\Big)\,dz\,dx.\end{aligned}$$

Therefore, for any $f,\, g\in B_b(\mathbb{R}^d)$,

$$\begin{aligned}&-\int f(x)L_{>1}g(x)\,\mu_V(dx)\\ &=\frac14\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{(f(x+ze_i)-f(x))(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}\\ &\qquad\qquad\times\Big(e^{-V(x)}+e^{-V(x+ze_i)}\Big)\,dz\,dx\\ &=\frac14\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{(f(x+ze_i)-f(x))(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}e^{-V(x)}\,dz\,dx\\ &\quad+\frac14\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{(f(x+ze_i)-f(x))(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\,dx\\ &=\frac12\sum_{i=1}^d\int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}}\frac{(f(x+ze_i)-f(x))(g(x+ze_i)-g(x))}{|z|^{1+\alpha}}\,dz\,\mu_V(dx)\end{aligned}$$

$$= D_{>1}(f, g),$$

where the third equality follows from the change of variables $z \to -z$ and $x \to x+ze_i$ again in the second term of the second equality.

(2) By the definition of $\mathscr{C}_\gamma$, it is easy to see that $B_b(\mathbb{R}^d) \subset \mathscr{C}_\gamma$. For any $g \in \mathscr{C}_\gamma$,

$$\begin{aligned}
|L_{>1}g(x)| \leqslant & \frac{1}{2}\big(1 + \|e^{-V}\|_\infty e^{V(x)}\big) \sum_{i=1}^d \int_{\{|z|>1\}} \big|g(x+ze_i) - g(x)\big| \frac{1}{|z|^{1+\alpha}}\, dz \\
\leqslant & \frac{dC}{\alpha-\gamma}\big(1 + \|e^{-V}\|_\infty e^{V(x)}\big),
\end{aligned}$$

from which we know that $L_{>1}g(x)$ is a well-defined and locally bounded function on $\mathbb{R}^d$. On the other hand,

$$\begin{aligned}
&\sum_{i=1}^d \int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}} \frac{|f(x+ze_i) - f(x)||g(x+ze_i) - g(x)|}{|z|^{1+\alpha}}\, dz\, \mu_V(dx) \\
&\leqslant C \sum_{i=1}^d \int_{\{\mathbb{R}^d\times\mathbb{R}:|z|>1\}} \frac{|f(x+ze_i) - f(x)|}{|z|^{1+\alpha-\gamma}}\, dz\, \mu_V(dx) \\
&\leqslant \frac{4Cd\|f\|_\infty}{\alpha-\gamma}.
\end{aligned}$$

Then the last assertion follows from the argument in part (1). □

Remark 2.2. According to Proposition 2.1(1), the operator $L_{>1}$ is a (formal) generator of the Dirichlet form $(D_{>1}, \mathscr{D}(D_>1))$. However, to verify that the operator $L_{>1}$ maps $B_b(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d;\mu_V)$, we need some additional assumption. For sufficient conditions that $L_{>1}$ maps $C_c^\infty(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d;\mu_V)$, one can refer to [5, Theorem 2.1].

Let $\gamma \in (0, 1\wedge\alpha)$ and define

$$\phi(x) = 1 + \sum_{i=1}^d |x_i|^\gamma,$$

which is the Lyapunov function we mentioned in Remark (3) in the end of Section 1.1. Then, for any $x, y \in \mathbb{R}^d$,

$$|\phi(x) - \phi(y)| \leqslant \sum_{i=1}^d \big||x_i|^\gamma - |y_i|^\gamma\big| \leqslant \sum_{i=1}^d |x_i - y_i|^\gamma \leqslant d|x-y|^\gamma,$$

and so $\phi \in \mathscr{C}_\gamma$. It follows from Proposition 2.1(2) that $L_{>1}\phi(x)$ is a well-defined locally bounded function on $\mathbb{R}^d$. Indeed, we have the following explicit estimate for $L_{>1}\phi(x)$.

Proposition 2.3. *For any $x \in \mathbb{R}^d$, let $\Gamma^V_{\inf}(x)$, $\Gamma^V_{\sup}(x)$ and $\Lambda(x)$ be these defined in Theorem 1.1. Suppose that (1.3) holds with some $\gamma \in (0, \alpha\wedge 1)$, and*

$$\liminf_{|x|\to\infty} \frac{e^{V(x)}\Gamma^V_{\inf}(x)}{|x|^{1+\alpha}} > 0. \tag{2.9}$$

Then, there exist constants C_1, C_2 and $r_0 > 0$ such that for all $x \in \mathbb{R}^d$,

$$L_{>1}\phi(x) \leqslant -C_1\Lambda(x)\phi(x)\mathbb{1}_{B(0,r_0)^c}(x) + C_2\mathbb{1}_{B(0,r_0)}(x). \tag{2.10}$$

Proof. According to Proposition 2.1(2), we only need to verify (2.10) for $|x|$ large enough. First, by the fact that for any a, $b \in \mathbb{R}$, $|a+b|^\gamma \leqslant |a|^\gamma + |b|^\gamma$, we get that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \big(\phi(x+ze_i)-\phi(x)\big)\frac{1}{|z|^{1+\alpha}}\,dz &\leqslant \frac{1}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \frac{1}{|z|^{1+\alpha-\gamma}}\,dz\\ &= \frac{d}{\alpha-\gamma}.\end{aligned}$$

On the other hand, for $|x| > 2^{1/\gamma}\sqrt{d}$ large enough, there is an integer $1 \leqslant k \leqslant d$ such that $|x_k| \geqslant |x|/\sqrt{d}$, and so

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \big(\phi(x+ze_i)-\phi(x)\big)\frac{e^{V(x)-V(x+ze_i)}}{|z|^{1+\alpha}}\,dz\\
&= \frac{e^{V(x)}}{2}\sum_{i=1}^d \int_{\{|z|>1\}} \frac{|x_i+z|^\gamma-|x_i|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\\
&= \frac{e^{V(x)}}{2}\bigg[\sum_{i=1}^d \int_{\{|x_i+z|\leqslant|x_i|,|z|>1\}} \frac{|x_i+z|^\gamma-|x_i|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\\
&\qquad + \sum_{i=1}^d \int_{\{|x_i+z|\geqslant|x_i|,|z|>1\}} \frac{|x_i+z|^\gamma-|x_i|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\bigg]\\
&\leqslant \frac{e^{V(x)}}{2}\bigg[\int_{\{|x_k+z|\leqslant|x_k|,|z|>1\}} \frac{|x_k+z|^\gamma-|x_k|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_k)}\,dz\\
&\qquad + \sum_{i=1}^d \int_{\{|x_i+z|\geqslant|x_i|,|z|>1\}} \frac{|x_i+z|^\gamma-|x_i|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\bigg],\end{aligned}$$

where in the inequality above we have dropped the sum with $i \neq k$, since it is negative. It is easy to see that the right hand side is dominated by

$$\begin{aligned}&\frac{e^{V(x)}}{2}\bigg[\int_{\{|x_k+z|\leqslant 1,|z|>1\}} \frac{1-|x_k|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_k)}\,dz\\
&\qquad + \int_{\{1<|x_k+z|\leqslant|x_k|,|z|>1\}} \frac{|x_k+z|^\gamma-|x_k|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_k)}\,dz\\
&\qquad + \sum_{i=1}^d \int_{\{|x_i+z|\geqslant|x_i|,|z|>1\}} \frac{|x_i+z|^\gamma-|x_i|^\gamma}{|z|^{1+\alpha}}e^{-V(x+ze_i)}\,dz\bigg]\\
&\leqslant -\frac{e^{V(x)}}{4}\bigg(\Big(\inf_{|u_k|\leqslant 1} e^{-V(x_1,\cdots,x_{k-1},u_k,x_{k+1},\cdots,x_d)}\Big)\int_{\{|x_k+z|\leqslant 1\}}\frac{dz}{|z|^{1+\alpha}}\bigg)|x_k|^\gamma\\
&\quad + \frac{de^{V(x)}}{2}\Gamma^V_{\sup}(x)\int_{\{|z|>1\}}\frac{|z|^\gamma}{|z|^{1+\alpha}}\,dz,\end{aligned}$$

where in the inequality above we have used the fact that for any x_k, $z \in \mathbb{R}$ with $|x_k| \geqslant |x|/\sqrt{d} \geqslant 2^{1/\gamma} > 2$ and $|x_k+z| \leqslant 1$, it holds $|z| \geqslant |x_k| - |x_k+z| > 1$, and we also have dropped the second term since it is negative too. Furthermore, combining

the fact $|x_k| \geqslant |x|/\sqrt{d}$ with (2.9) and (1.3), we find that the right hand side of the inequality above is smaller than

$$-c_1\frac{e^{V(x)}}{(1+|x|)^{1+\alpha}}\Gamma^V_{\inf}(x)\phi(x)+c_2e^{V(x)}\Gamma^V_{\sup}(x)\leqslant -c_3\frac{e^{V(x)}}{(1+|x|)^{1+\alpha}}\Gamma^V_{\inf}(x)\phi(x),$$

where $c_i(i=1,2,3)$ are positive constants. Therefore, the desired assertion follows from all the estimates above. □

Next, we turn to the local super Poincaré inequality for $(D,\mathscr{D}(D))$. As mentioned in Section 1, the local super Poincaré inequality for $(D_S,\mathscr{D}(D_S))$ is derived from the classical fractional Sobolev inequality, see [17, Lemma 3.1]. However, it seems that the Dirichlet form $(D,\mathscr{D}(D))$ does not have a connection with the fractional Laplacian $-(-\Delta)^{\alpha/2}$. Thus, in the following proposition we need a completely different approach, which is connected with the property of doubling measure in harmonic analysis, see [12].

Proposition 2.4. *There exists a constant $C_3>0$ such that for any $f\in C_c^\infty(\mathbb{R}^d)$ and $r>0$,*

$$\int_{B(0,r)} f^2(x)\,\mu_V(dx)\leqslant tD(f,f)+\beta_r(t)\mu_V(|f|)^2,\quad t>0, \tag{2.11}$$

where

$$\beta_r(t)=C_3\left[t\wedge\left(r^\alpha\frac{\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}}{\inf_{|x|\leqslant r}e^{V(x)}}\right)\right]^{-d/\alpha}\frac{\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)^{2+d/\alpha}}{\left(\inf_{|x|\leqslant r}e^{V(x)}\right)^{1+d/\alpha}}.$$

In particular, for any $r_0>0$, there is a constant C_4 depending on r_0 such that for any $f\in C_c^\infty(\mathbb{R}^d)$ and $r\geqslant r_0$, the local Poincaré inequality (2.11) *holds with*

$$\beta_r(t)=C_4\left(1+t^{-d/\alpha}\right)\frac{\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)^{2+d/\alpha}}{\left(\inf_{|x|\leqslant r}e^{V(x)}\right)^{1+d/\alpha}}$$

for all $t>0$.

Proof. The second assertion immediately follows from the first one, by the fact that the function $r\mapsto r^\alpha\frac{\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}}{\inf_{|x|\leqslant r}e^{V(x)}}$ is increasing. Thus, we only need to prove the first one, which is split into three steps.

(1) For any $0<s\leqslant r$ and $f\in C_c^\infty(\mathbb{R}^d)$, define

$$f_s(x):=\frac{1}{|B(0,s)|}\int_{B(x,s)}f(z)\,dz,\quad x\in B(0,r),$$

where $|B(0,s)|$ denotes the volume of the ball with center at 0 and radius s. We have

$$\sup_{x\in B(0,r)}|f_s(x)|\leqslant\frac{1}{|B(0,s)|}\int_{B(0,2r)}|f(z)|\,dz,$$

and

$$\begin{aligned}\int_{B(0,r)}|f_s(x)|\,dx&\leqslant\int_{B(0,r)}\frac{1}{|B(0,s)|}\int_{B(x,s)}|f(z)|\,dz\,dx\\&\leqslant\int_{B(0,2r)}\left(\frac{1}{|B(0,s)|}\int_{B(z,s)}dx\right)|f(z)|\,dz\end{aligned}$$

$$\leqslant \int_{B(0,2r)} |f(z)|\,dz.$$

Thus,

$$\begin{aligned}\int_{B(0,r)} f_s^2(x)\,dx \leqslant&\Big(\sup_{x\in B(0,r)} |f_s(x)|\Big)\int_{B(0,r)} |f_s(x)|\,dx\\ \leqslant&\frac{1}{|B(0,s)|}\bigg(\int_{B(0,2r)} |f(z)|\,dz\bigg)^2.\end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality, for any $f \in C_c^\infty(\mathbb{R}^d)$ and $0 < s \leqslant r$,

$$\begin{aligned}&\int_{B(0,r)} f^2(x)\,dx\\ &\quad\leqslant 2\int_{B(0,r)} \big(f(x)-f_s(x)\big)^2\,dx + 2\int_{B(0,r)} f_s^2(x)\,dx\\ &\quad\leqslant 2\int_{B(0,r)} \frac{1}{|B(0,s)|}\int_{B(x,s)} (f(x)-f(y))^2\,dy\,dx + \frac{2}{|B(0,s)|}\bigg(\int_{B(0,2r)} |f(z)|\,dz\bigg)^2.\end{aligned}$$

Using the convention that $(y_0 - x_0)e_0 = 0$ and the inequality that

$$(a_1 + \ldots + a_d)^2 \leqslant d(a_1^2 + \ldots + a_d^2), \quad a_1,\ldots,a_d \in \mathbb{R}$$

deduced from the Cauchy-Schwarz inequality again, we find that the right hand side is dominated by

$$\begin{aligned}&\frac{2d}{|B(0,s)|}\sum_{i=1}^d \int_{B(0,r)}\int_{B(x,s)} \Big(f\big(x+(y_1-x_1)e_1+\ldots+(y_i-x_i)e_i\big)\\ &\qquad\qquad - f\big(x+(y_1-x_1)e_1+\ldots+(y_{i-1}-x_{i-1})e_{i-1}\big)\Big)^2\,dy\,dx\\ &\quad + \frac{2}{|B(0,s)|}\bigg(\int_{B(0,2r)} |f(z)|\,dz\bigg)^2\\ &\leqslant \frac{2d}{|B(0,s)|}\sum_{i=1}^d \int_{B(0,r)}\int_{\{|z_1|\leqslant s\}}\cdots\int_{\{|z_d|\leqslant s\}} \Big(f\big(x+z_1e_1+\ldots+z_ie_i\big)\\ &\qquad\qquad - f\big(x+z_1e_1+\ldots+z_{i-1}e_{i-1}\big)\Big)^2\,dz_1\ldots dz_d\,dx\\ &\quad + \frac{2}{|B(0,s)|}\bigg(\int_{B(0,2r)} |f(z)|\,dz\bigg)^2\\ &\leqslant \frac{2^d d s^{d-1}}{|B(0,s)|}\sum_{i=1}^d \int_{B(0,\sqrt{d}(r+s))}\int_{\{|z_i|\leqslant s\}} \Big(f\big(x+z_ie_i\big)-f(x)\Big)^2\,dz_i\,dx\\ &\quad + \frac{2}{|B(0,s)|}\bigg(\int_{B(0,2r)} |f(z)|\,dz\bigg)^2,\end{aligned}$$

where the two inequalities above follow from setting $z_k = y_k - z_k$ for all $1 \leqslant k \leqslant i-1$ and enlarging the domain of x respectively. On the other hand, we can easily see

that the right hand side of the inequality above is smaller than

$$\frac{2^d d s^{d+\alpha}}{|B(0,s)|}\sum_{i=1}^d \int_{B(0,\sqrt{d}(r+s))}\int_{\{|z|\leqslant s\}} \frac{\Big(f\big(x+ze_i\big)-f(x)\Big)^2}{|z|^{1+\alpha}}\,dz\,dx$$
$$+\frac{2}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,dz\right)^2.$$

Therefore, for any $f\in C_c^\infty(\mathbb{R}^d)$ and $0<s\leqslant r$,

$$\int_{B(0,r)} f^2(x)\,dx \leqslant \frac{2^d d s^{d+\alpha}}{|B(0,s)|}\sum_{i=1}^d \int_{B(0,\sqrt{d}(r+s))}\int_{\{|z|\leqslant s\}} \frac{\Big(f\big(x+ze_i\big)-f(x)\Big)^2}{|z|^{1+\alpha}}\,dz\,dx$$
$$+\frac{2}{|B(0,s)|}\left(\int_{B(0,2r)}|f(z)|\,dz\right)^2.$$

(2) According to the inequality above, for any $f\in C_c^\infty(\mathbb{R}^d)$ and $0<s\leqslant r$,

$$\begin{aligned}
&\int_{B(0,r)} f^2(x)\,\mu_V(dx)\\
&\leqslant \frac{1}{\inf_{|x|\leqslant r} e^{V(x)}}\int_{B(0,r)} f^2(x)\,dx\\
&\leqslant \left[\frac{2^d d s^{d+\alpha}}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\right]\sum_{i=1}^d \int_{B(0,\sqrt{d}(r+s))}\int_{\{|z|\leqslant s\}} \frac{\Big(f\big(x+ze_i\big)-f(x)\Big)^2}{|z|^{1+\alpha}}\,dz\,dx\\
&\quad+\frac{2}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\left(\int_{B(0,2r)}|f(z)|\,dz\right)^2\\
&\leqslant \left[\frac{2^d d s^{d+\alpha}\big(\sup_{|x|\leqslant 2\sqrt{d}r} e^{V(x)}\big)}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\right]\\
&\qquad\qquad\times\sum_{i=1}^d \int_{B(0,2\sqrt{d}r)}\int_{\{|z|\leqslant s\}} \frac{\Big(f\big(x+ze_i\big)-f(x)\Big)^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx)\\
&\quad+\frac{2\big(\sup_{|x|\leqslant 2r} e^{V(x)}\big)^2}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\left(\int_{B(0,2r)}|f(x)|\,\mu_V(dx)\right)^2\\
&\leqslant \left[\frac{2^{d+1} d s^{d+\alpha}\big(\sup_{|x|\leqslant 2\sqrt{d}r} e^{V(x)}\big)}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\right] D(f,f)+\frac{2\big(\sup_{|x|\leqslant 2r} e^{V(x)}\big)^2}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\mu_V(|f|)^2,
\end{aligned}$$

which implies that the local super Poincaré inequality (2.11) holds with

$$\beta_r(t)=\inf\left\{\frac{2\big(\sup_{|x|\leqslant 2r} e^{V(x)}\big)^2}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}: 0<s\leqslant r \text{ and } \frac{2^{d+1} d s^{d+\alpha}\big(\sup_{|x|\leqslant 2\sqrt{d}r} e^{V(x)}\big)}{|B(0,s)|\big(\inf_{|x|\leqslant r} e^{V(x)}\big)}\leqslant t\right\}$$

for any $t>0$.

(3) Next, we fix $r>0$. If $0<t\leqslant \frac{2^{d+1}dr^{d+\alpha}\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)}{|B(0,r)|\left(\inf_{|x|\leqslant r}e^{V(x)}\right)}$, then one can choose $s\in(0,r]$ such that $\frac{2^{d+1}ds^{d+\alpha}\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)}{|B(0,s)|\left(\inf_{|x|\leqslant r}e^{V(x)}\right)}=t$, and so there is a constant $C_5>0$ (independent of r,t) such that $\beta_r(t)\leqslant C_5t^{-d/\alpha}\frac{\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)^{2+d/\alpha}}{\left(\inf_{|x|\leqslant r}e^{V(x)}\right)^{1+d/\alpha}}$.

If $t\geqslant \frac{2^{d+1}dr^{d+\alpha}\left(\sup_{|x|\leqslant 2\sqrt{d}r}e^{V(x)}\right)}{|B(0,r)|\left(\inf_{|x|\leqslant r}e^{V(x)}\right)}$, then, by taking $s=r$ in the right hand side of the definition of $\beta_r(t)$ above, we find that $\beta_r(t)\leqslant C_6\frac{\left(\sup_{|x|\leqslant 2r}e^{V(x)}\right)^2}{r^d\left(\inf_{|x|\leqslant r}e^{V(x)}\right)}$.

Combining with both estimates above, we complete the proof. □

3. Proofs and Complements

3.1. **Proofs of Theorem 1.1, Corollaries and Example.** We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $C_c^\infty(\mathbb{R}^d)$ is the core of $\mathscr{D}(D)$, it is enough to prove the desired Poincaré type inequalities for any $f\in C_c^\infty(\mathbb{R}^d)$. According to Proposition 2.3 and $\phi\geqslant 1$, there exists a constant $r_0>0$ such that

$$\mathbb{1}_{B(0,r)^c}\leqslant \frac{1}{C_1\Phi(r)}\frac{-L_{>1}\phi}{\phi}+\frac{C_2}{C_1\Phi(r)}\mathbb{1}_{B(0,r_0)},\quad r\geqslant r_0,$$

where

$$\Phi(r)=\inf_{|x|\geqslant r}\Lambda(x).$$

Then, for any $f\in C_c^\infty(\mathbb{R}^d)$,

$$\mu_V(f^2\mathbb{1}_{B(0,r)^c})\leqslant \frac{1}{C_1\Phi(r)}\mu_V\left(f^2\frac{-L_{>1}\phi}{\phi}\right)+\frac{C_2}{C_1\Phi(r)}\mu_V(f^2\mathbb{1}_{B(0,r_0)}),\quad r\geqslant r_0.$$

We note that for any $x,\ y\in\mathbb{R}^d$,

$$\begin{aligned}\left(\frac{f^2(x)}{\phi(x)}-\frac{f^2(y)}{\phi(y)}\right)(\phi(x)-\phi(y))&=f^2(x)+f^2(y)-\left(\frac{\phi(y)}{\phi(x)}f^2(x)+\frac{\phi(x)}{\phi(y)}f^2(y)\right)\\&\leqslant f^2(x)+f^2(y)-2|f(x)||f(y)|\\&\leqslant (f(x)-f(y))^2,\end{aligned}$$

which, along with Proposition 2.1(2), yields that

$$\mu_V\left(f^2\frac{-L_{>1}\phi}{\phi}\right)\leqslant D_{>1}(f,f)\leqslant D(f,f).$$

Therefore, for any $f\in C_c^\infty(\mathbb{R}^d)$,

$$\mu_V(f^2\mathbb{1}_{B(0,r)^c})\leqslant \frac{1}{C_1\Phi(r)}D(f,f)+\frac{C_2}{C_1\Phi(r)}\mu_V(f^2\mathbb{1}_{B(0,r_0)}),\quad r\geqslant r_0.\tag{3.12}$$

On the other hand, by Proposition 2.4, there exists a constant $C_3>0$ such that for any $f\in C_c^\infty(\mathbb{R}^d)$, $r\geqslant r_0$ and $t>0$,

$$\mu_V(f^2\mathbb{1}_{B(0,r)})\leqslant tD(f,f)+\beta_r(t)\mu_V(|f|)^2,$$

where

$$\beta_r(t) = C_3(1+t^{-d/\alpha})\frac{\left(\sup_{|x|\leqslant 2\sqrt{d}r} e^{V(x)}\right)^{2+d/\alpha}}{\left(\inf_{|x|\leqslant r} e^{V(x)}\right)^{1+d/\alpha}}.$$

Combining it with (3.12), we get that for any $f \in C_c^\infty(\mathbb{R}^d)$, $r \geqslant r_0$ and $t>0$,

$$\begin{aligned}\mu_V(f^2) =&\mu_V(f^2\mathbb{1}_{B(0,r)^c}) + \mu_V(f^2\mathbb{1}_{B(0,r)})\\ \leqslant& \left(t+\frac{1+C_2t}{C_1\Phi(r)}\right) D(f,f) + \beta_r(t)\left(1+\frac{C_2}{C_1\Phi(r)}\right)\mu_V(|f|)^2.\end{aligned} \tag{3.13}$$

(1) Suppose that

$$\lim_{r\to\infty}\Phi(r) > 0.$$

Then, for any fixed $t>0$, (3.13) is just the defective Poincaré inequality. Since the conservative and symmetric Dirichlet form $(D,\mathscr{D}(D))$ is irreducible, i.e. $D(f,f)=0$ implies f is a constant function, it follows from [15, Corollary 1.2] (see also [11, Theorem 1]) that the defective Poincaré inequality (3.13) implies the true Poincaré inequality (1.4).

(2) Now, we assume that

$$\lim_{r\to\infty}\Phi(r) = \infty.$$

For any $s>0$, taking $t = s/2$ and $r = \Phi^{-1}((2+C_2s)/(C_1s))$ in (3.13), we can get the super Poincaré inequality (1.5) with the desired rate function β (possibly with different positive constants C_1 and C_2). $\square$

Proof of Corollary 1.2. Let $\varepsilon_* = \min_{1\leqslant i\leqslant d}\varepsilon_i$. We have

$$\Gamma^V_{\inf}(x) \geqslant c_1e^{-V(x)}(1+|x|)^{1+\varepsilon_*},\ \Gamma^V_{\sup}(x) = e^{-V(x)} \text{ and } \Lambda(x) \geqslant c_2(1+|x|)^{\varepsilon_*-\alpha}$$

for some constants $c_1, c_2 > 0$. Then, (1.3) is satisfied for any $\gamma>0$, if $\varepsilon_*\geqslant\alpha$. Next, we will prove the desired assertions.

(1) If $\varepsilon_i \geqslant \alpha$ for all $1\leqslant i\leqslant d$, then the Poincaré inequality (1.4) follows from Theorem 1.1(1). Assume that for some $1\leqslant i\leqslant d$, $\varepsilon_i\in(0,\alpha)$. To disprove the Poincaré inequality (1.4) in this case, let us consider the function $f(x) = g(x_i)$, where $g\in C_c^\infty(\mathbb{R})$. By [5, Theorem 2.1(1)], in the present setting $C_1^b(\mathbb{R}^d)\subset\mathscr{D}(D)$, and so we can apply $f\in C_b^\infty(\mathbb{R}^d)\subset\mathscr{D}(D)$ into the Poincaré inequality (1.4). Furthermore, it is easy to see that for this class of functions, the Poincaré inequality (1.4) is reduced into

$$m(g^2)\leqslant C\int_{\mathbb{R}\times\mathbb{R}}\frac{(g(x+z)-g(x))^2}{|z|^{1+\alpha}}\,dz\,m(dx),\quad m(g)=0, g\in C_c^\infty(\mathbb{R}), \tag{3.14}$$

where $m(dx) = C_{\varepsilon_i}(1+|x|)^{-(1+\varepsilon_i)}\,dx$. According to [17, Corollary 1.2], we know that for $\varepsilon_i\in(0,\alpha)$, the inequality (3.14) does not hold. Therefore, for any constant $C>0$, the Poincaré inequality (1.4) also does not hold.

(2) Let $\varepsilon_i>\alpha$ for all $1\leqslant i\leqslant d$. It is easy to see that $\Phi(r)\geqslant c_3r^{\varepsilon_*-\alpha}$ for r large enough. Then, $\Phi^{-1}(r^{-1})\leqslant c_4r^{-1/(\varepsilon_*-\alpha)}$ for r small enough, so that

$$\beta(r)\leqslant c_5\left(1+r^{-(\frac{d}{\alpha}+\frac{(2\alpha+d)\sum_{i=1}^d(1+\varepsilon_i)}{\alpha(\varepsilon_*-\alpha)})}\right),\quad r>0$$

for some constant $c_5>0$. The equivalence of the super Poincaré inequality and the corresponding bound of $\|P_t\|_{L^1(\mathbb{R}^d;\mu_V)\to L^\infty(\mathbb{R}^d;\mu_V)}$ then follows from [16, Theorem 3.3.15(2)].

Next, we prove that if $\varepsilon_i \in (0, \alpha]$ for some $1 \leqslant i \leqslant d$, then for any $\beta : (0, \infty) \to (0, \infty)$ the super Poincaré inequality (1.5) does not hold. Indeed, if the inequality (1.5) holds, then, applying the function $f(x) = g(x_i)$ (where $g \in C_c^\infty(\mathbb{R})$) mentioned above, we have

$$m(g^2) \leqslant r \int_{\mathbb{R}\times\mathbb{R}} \frac{(g(x+z) - g(x))^2}{|z|^{1+\alpha}} \, dz \, m(dx) + \beta(r) m(|g|)^2, \quad r > 0, g \in C_c^\infty(\mathbb{R}). \tag{3.15}$$

However, according to [17, Corollary 1.2(2)], (3.15) can not be true. □

Proof of Corollary 1.3. Suppose that $\varepsilon_* = \min_{1\leqslant i\leqslant d} \varepsilon_i \geqslant 0$. Then, there are constants $c_i > 0$ $(i = 1, 2)$ such that for all $x \in \mathbb{R}^d$,

$$\Gamma^V_{\inf}(x) \geqslant c_1 e^{-V(x)}(1 + |x|)^{1+\alpha} \log^{\varepsilon_*}(e + |x|), \ \Gamma^V_{\sup}(x) = e^{-V(x)}$$

and $\Lambda(x) \geqslant c_2 \log^{\varepsilon_*}(e + |x|)$. It is obvious that (1.3) holds.

(1) If $\varepsilon_i \geqslant 0$ for all $1 \leqslant i \leqslant d$, then the Poincaré inequality (1.4) follows from Theorem 1.1(1). To prove that the Poincaré inequality (1.4) does not hold when $\varepsilon_i < 0$ for some $1 \leqslant i \leqslant d$, we only need to consider the Poincaré inequality (3.14), where $m(dx) = C_{\varepsilon_i}(1+|x|)^{-(1+\alpha)} \log^{-\varepsilon_i}(e+|x|)\, dx$. Then, according to [17, Corollary 1.3], (3.14) does not hold, and so (1.4) does not too.

(2) Let $\varepsilon_i > 0$ for all $1 \leqslant i \leqslant d$. It is easy to see that $\Phi(r) \geqslant c_3 \log^{\varepsilon_*} r$ for r large enough. Then, $\Phi^{-1}(1/r) \leqslant c_4 \exp(c_4 r^{-1/\varepsilon_*})$ for r small enough, so that

$$\beta(r) \leqslant \exp\left(c_5(1 + r^{-1/\varepsilon_*})\right), \quad r > 0$$

for some constant $c_5 > 0$. When $\varepsilon_i > 1$ for all $1 \leqslant i \leqslant d$, the equivalence of the super Poincaré inequality and the corresponding bound of $\|P_t\|_{L^1(\mathbb{R}^d;\mu_V)\to L^\infty(\mathbb{R}^d;\mu_V)}$ follows from [16, Theorem 3.3.15(1)].

Next, we prove that if $\varepsilon_i \leqslant 0$ for some $1 \leqslant i \leqslant d$, then for any $\beta : (0, \infty) \to (0, \infty)$ the super Poincaré inequality (1.5) does not hold. Indeed, in this case, we only need to prove that the super Poincaré inequality (3.15) does not hold for $m(dx) = C_{\varepsilon_i}(1 + |x|)^{-(1+\alpha)} \log^{-\varepsilon_i}(e + |x|)\, dx$. This is just a consequence of [17, Corollary 1.3].

According to [16, Corollary 3.3.4(1)], the super Poincaré inequality with $\beta(r) = \exp(c(1 + r^{-1}))$ for some $c > 0$ is equivalent to the log-Sobolev inequality for some constant $C > 0$, and so according to the conclusions above we can conclude the last assertion in (2) for log-Sobolev inequality. □

Proof of Example 1.4. We first estimate $\Gamma^V_{\inf}(x)$ and $\Gamma^V_{\sup}(x)$ respectively. On the one hand, for all $x \in \mathbb{R}^d$, by using the boundness of a_i for all $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\Gamma^V_{\inf}(x) \geqslant & c_1 \inf_{1\leqslant i\leqslant d: |x_i|\geqslant |x|/\sqrt{d}} \inf_{|u_i|\leqslant 1} \left[(1 + |u_i|)^{-1-a_i^*(x|u_i)} \prod_{j\neq i} (1 + |x_j|)^{-1-a_j^*(x|u_i)} \right] \\
\geqslant & c_2 \inf_{1\leqslant i\leqslant d: |x_i|\geqslant |x|/\sqrt{d}} \inf_{|u_i|\leqslant 1} \left[\prod_{j\neq i} (1 + |x_j|)^{-1-a_j^*(x|u_i)} \right] \\
\geqslant & c_2 \inf_{1\leqslant i\leqslant d: |x_i|\geqslant |x|/\sqrt{d}} \left[\prod_{j\neq i} (1 + |x_j|)^{-1-\sup_{|u_i|\leqslant 1} a_j^*(x|u_i)} \right] \\
= & c_2 \inf_{1\leqslant i\leqslant d: |x_i|\geqslant |x|/\sqrt{d}} \left[(1 + |x_i|)^{1+\sup_{|u_i|\leqslant 1} a_i^*(x|u_i)} \prod_{j=1}^{d} (1 + |x_j|)^{-1-\sup_{|u_i|\leqslant 1} a_j^*(x|u_i)} \right]
\end{aligned}$$

$$\geqslant c_3(1+|x|)^{1+A(x)}\prod_{j=1}^{d}(1+|x_j|)^{-1-N_j(x)}.$$

On the other hand, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\Gamma^V_{\sup}(x) \leqslant & c_4 \sup_{1\leqslant i\leqslant d}\sup_{|u_i|\geqslant|x_i|}\left[(1+|u_i|)^{-1-a_i^*(x|u_i)}\prod_{j\neq i}(1+|x_j|)^{-1-a_j^*(x|u_i)}\right]\\ \leqslant & c_4\sup_{1\leqslant i\leqslant d}\prod_{1\leqslant j\leqslant d}(1+|x_j|)^{-1-\inf_{|u_i|\geqslant|x_i|}a_j^*(x|u_i)}\\ \leqslant & c_4\prod_{1\leqslant j\leqslant d}(1+|x_j|)^{-1-M_j(x)}.\end{aligned}$$

Under (1.6) and (1.7), (1.3) holds true for any $\gamma > 0$. Furthermore, (1.6) implies that for $|x|$ large enough and all $1 \leqslant j \leqslant d$, $a_j(x) \geqslant N_j(x)$, so for $|x|$ large enough,

$$\Lambda(x) \geqslant c_5(1+|x|)^{A(x)-\alpha}.$$

Having these estimates at hand, we can obtain the required assertions by following the arguments of Corollary 1.2 and using Theorem 1.1. □

3.2. **Complement: Entropy Inequalities and Tensorisation Property for Singular Stable-like Dirichlet Forms.** In this part, we are concerned with the case that the reference measure μ_V is a product measure on $\mathbb{R}^d$, and aim to consider entropy inequalities for the Dirichlet form $(D, \mathscr{D}(D))$. Note that, the relative entropy Ent_μ is defined on $L^1(\mathbb{R}^d;\mu)$ as follows

$$\mathrm{Ent}_\mu(f) := \mu(f\log f) - \mu(f)\log\mu(f), \quad f>0.$$

The following theorem is a direct application of [18, Theorem 1.5] and the sub-additivity property of the relative entropy.

Theorem 3.1. *Let $\mu_V = \mu_1\otimes\ldots\otimes\mu_d$ be a product measure on $\mathbb{R}^d$ such that for any $1\leqslant i\leqslant d$, $\mu_i(dx_i) = e^{-V_i(x_i)}\,dx_i$ is a probability measure on $\mathbb{R}$, where V_i is a Borel measurable function on $\mathbb{R}$ and $e^{-V_i(\cdot)}$ may be unbounded. If for any $1\leqslant i\leqslant d$ there exists a constant $C_i>0$ such that for any $x,y\in\mathbb{R}$*

$$\frac{e^{V_i(x)}+e^{V_i(y)}}{|x-y|^{1+\alpha}} \geqslant C_i, \tag{3.16}$$

then the following entropy inequality holds

$$\mathrm{Ent}_{\mu_V}(f) \leqslant 2\big(\sup_{1\leqslant i\leqslant d} C_i^{-1}\big)D(f,\log f), \quad f\in\mathscr{D}(D), f>0. \tag{3.17}$$

In particular, the Poincaré inequality (1.4) *also holds.*

Proof. For the sake of completeness, we provide the details here. For any $f\in\mathscr{D}(D)$ with $f>0$ and for any $x\in\mathbb{R}^d$, define

$$f_{x,i}(y_i) := f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_d), \quad y_i\in\mathbb{R}, 1\leqslant i\leqslant d.$$

By the sub-additivity property of the relative entropy (see [10, Proposition 4.1] or [9, Corollary 3]), for any $f\in\mathscr{D}(D)$ with $f>0$,

$$\mathrm{Ent}_{\mu_V}(f) \leqslant \sum_{i=1}^{d}\int \mathrm{Ent}_{\mu_i} f_{x,i}(y_i)\prod_{j\neq i}\mu_j(dx_j).$$

According to (3.16) and [18, Theorem 1.5], for each i the inner relative entropy is at most

$$C_i^{-1}\int_{\mathbb{R}\times\mathbb{R}}\frac{\big(f(x+ze_i)-f(x)\big)\big(\log f(x+ze_i)-\log f(x)\big)}{|z|^{1+\alpha}}\,dz\,\mu_i(dx_i).$$

Indeed, write $f_{x,i}$ as f_i for simplicity. By the Jensen inequality,

$$\begin{aligned}\mathrm{Ent}_{\mu_i}(f_i)&=\mu_i(f_i\log f_i)-\mu_i(f_i)\log\mu_i(f_i)\\&\leqslant \mu_i(f_i\log f_i)-\mu_i(f_i)\mu_i(\log f_i)\\&=\frac12\int\Big[f_i(x_i)\log f_i(x_i)+f_i(y_i)\log f_i(y_i)\\&\qquad -f_i(x_i)\log f_i(y_i)-f_i(y_i)\log f_i(x_i)\Big]\,\mu_i(dy_i)\,\mu_i(dx_i)\\&=\frac12\int(f_i(x_i)-f_i(y_i))(\log f_i(x_i)-\log f_i(y_i))\,\mu_i(dy_i)\,\mu_i(dx_i).\end{aligned}\tag{3.18}$$

On the other hand, by (3.16), we find that

$$\begin{aligned}&\frac12\int\big(f_i(x_i)-f_i(y_i)\big)\big(\log f_i(x_i)-\log f_i(y_i)\big)\,\mu_i(dy_i)\,\mu_i(dx_i)\\&\leqslant C_i^{-1}\int_{x_i\neq y_i}\frac{\big(f_i(x_i)-f_i(y_i)\big)\big(\log f_i(x_i)-\log f_i(y_i)\big)}{|x_i-y_i|^{1+\alpha}}\,\frac{e^{-V_i(x_i)}+e^{-V_i(y_i)}}{2}\,dy_i\,dx_i\\&=C_i^{-1}\int_{\mathbb{R}\times\mathbb{R}}\frac{\big(f(x+ze_i)-f(x)\big)\big(\log f(x+ze_i)-\log f(x)\big)}{|z|^{1+\alpha}}\,dz\,\mu_i(dx_i),\end{aligned}$$

which, along with (3.18), yields the above desired assertion.

Summing up the conclusions above, we prove the required assertion for the entropy inequality (3.17). The last conclusion follows from the well known fact that the entropy inequality (3.17) is stronger than the Poincaré inequality (1.4). (To see this, one can apply (3.17) to the function $1+\varepsilon f$ and then take the limit as $\varepsilon\to 0$.) $\square$

The proof of Theorem 3.1 is based on the sub-additivity property of entropy and the characterization of Dirichlet form $(D,\mathscr{D}(D))$. Sub-additivity formulas are well known for variance and entropy, and then have been extended to Φ-entropies in [3, Proposition 3.1], which is called the tensorisation property. The tensorisation property for Φ-entropies also can be used to establish Φ-Sobolev inequalities for Dirichlet form $(D,\mathscr{D}(D))$. In particular, by tensorisation property the Φ-Sobolev inequalities are then infinite dimensional since they hold on the product space with the maximum of the one dimensional constants, e.g. see (3.17). However, such statements do not hold for weak/super Poincaré inequalities and transportation-cost inequalities. See [2, Theorem 5], [8, Section 1.3] and [14, Chapter 22] for more details.

To show that Theorem 3.1 is sharp, we consider the following corollary, which is regarded as a continuation of Corollary 1.2.

Corollary 3.2. *Let*

$$e^{-V(x)}=C_{\varepsilon_1,\dots,\varepsilon_d}\prod_{i=1}^d(1+|x_i|)^{-(1+\varepsilon_i)}$$

with $\varepsilon_i > 0$ for all $1 \leqslant i \leqslant d$. Then the entropy inequality (3.17) *holds for some constant $C > 0$ if and only if $\varepsilon_i \geqslant \alpha$ for all $1 \leqslant i \leqslant d$.*

Proof. If $\varepsilon_i \geqslant \alpha$ for all $1 \leqslant i \leqslant d$, then (3.16) holds (see [18, Example 1.6]), and so the desired entropy inequality (3.17) follows from Theorem 3.1. On the other hand, since the entropy inequality (3.17) is stronger than the Poincaré inequality (1.4), by Corollary 1.2(1), we know that (3.17) does not hold when $\varepsilon_i \in (0, \alpha)$ for some $1 \leqslant i \leqslant d$. □

3.3. **Complement: Weak Poincaré Inequalities for Singular Stable-like Dirichlet Forms.** In the end of this section, we turn to the weak Poincaré inequality, which can be used to characterize various convergence rates of the associated semigroups slower than exponential. In the following, let $\mu_{V_0}(dx) = e^{-V_0(x)}\,dx$ be a probability measure on $\mathbb{R}^d$ such that $e^{-V_0(x)}$ is a bounded measurable function, and (1.3) and $\lim_{r\to 0}\Phi(r) > 0$ hold with V_0 in place of V. Then, for a probability measure $\mu_V(dx) = e^{-V(x)}\,dx$, we have

Theorem 3.3. *Suppose that*

$$\sup_{x\in\mathbb{R}^d} e^{V(x)-V_0(x)} < \infty. \tag{3.19}$$

If there exist a family of Borel sets $\{A_s\}_{s\geqslant 0}$ such that $A_s \uparrow \mathbb{R}^d$ as $s \to \infty$, $\Phi_0(s) := \sup_{x\in A_s} e^{V_0(x)-V(x)} < \infty$ for any $s > 0$ and $\lim_{s\to\infty}\Phi_0(s) = \infty$, then the following weak Poincaré inequality

$$\mu_V(f^2) \leqslant \eta(r)D(f,f) + r\|f\|_\infty^2, \quad r > 0, f \in \mathscr{D}(D), \mu_V(f) = 0 \tag{3.20}$$

holds for

$$\eta(r) = C\inf\Big\{\Phi_0(s) : s > 0 \text{ such that } \mu_V(A_s) \geqslant \frac{1}{1+r}\Big\}$$

with some constant $C > 0$ independent of r.

In particular, under (3.19) *the weak Poincaré inequality* (3.20) *holds with*

$$\eta(r) = C_1\inf\Big\{s : s > 0 \text{ such that } \mu_V(D_s) \geqslant \frac{1}{1+r}\Big\},$$

where $D_s = \{x \in \mathbb{R}^d : e^{V_0(x)-V(x)} \leqslant s\}$ and $C_1 > 0$ is independent of r.

Proof. Since $C_c^\infty(\mathbb{R}^d)$ is a core of $\mathscr{D}(D)$, we only need to consider $f \in C_c^\infty(\mathbb{R}^d)$. According to Theorem 1.1(1), the following Poincaré inequality

$$\mu_{V_0}(f^2) \leqslant c_1 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}} \frac{(f(x+ze_i) - f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_{V_0}(dx),\ \ f \in C_c^\infty(\mathbb{R}^d), \mu_{V_0}(f) = 0$$

holds for some constant $c_1 > 0$. Let $\{A_s\}_{s\geqslant 0}$ be a family of subsets as in theorem. Therefore, for any $s > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{A_s}\left(f(x) - \frac{1}{\mu_V(A_s)}\int_{A_s} f(x)\,\mu_V(dx)\right)^2 \mu_V(dx)\\
&= \inf_{a\in\mathbb{R}}\int_{A_s}(f(x)-a)^2\,\mu_V(dx)\\
&\leqslant \int_{A_s}(f(x) - \mu_{V_0}(f))^2\,\mu_V(dx)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \left(\sup_{x\in A_s}\frac{e^{-V(x)}}{e^{-V_0(x)}}\right)\int_{A_s}(f(x)-\mu_{V_0}(f))^2\,\mu_{V_0}(dx)\\
&\leqslant c_1\left(\sup_{x\in A_s}\frac{e^{-V(x)}}{e^{-V_0(x)}}\right)\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_{V_0}(dx)\\
&\leqslant c_2\Phi_0(s)\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}}\frac{(f(x+ze_i)-f(x))^2}{|z|^{1+\alpha}}\,dz\,\mu_V(dx),
\end{aligned}$$

where in the last inequality we have used (3.19). Then the required weak Poincaré inequality (3.20) follows from [16, Theorem 4.3.1].

Taking $A_s=D_s$ and using the fact $\Phi_0(s)=\sup_{x\in D_s}e^{V_0(x)-V(x)}\leqslant s$, one can get the second assertion. □

We consider the following corollary to illustrate the power of Theorem 3.3.

Corollary 3.4. [Continuation of Corollaries 1.2 and 1.3]

(1) *Let*

$$e^{-V(x)}=C_{\varepsilon_1,\ldots,\varepsilon_d}\prod_{i=1}^d(1+|x_i|)^{-(1+\varepsilon_i)}$$

with $0<\varepsilon_*:=\min_{1\leqslant i\leqslant d}\varepsilon_i<\alpha$. *Then the weak Poincaré inequality* (3.20) *holds with*

$$\eta(r)=c_1\Big(1+r^{-\sum_{i=1}^d(\alpha-\varepsilon_i)^+/\varepsilon_*}\Big),\quad r>0 \tag{3.21}$$

for some constant $c_1>0$. *Consequently, there exists a constant* $\lambda>0$ *such that*

$$\begin{aligned}
\|P_t-\mu_V\|_{L^\infty(\mathbb{R}^d;\mu_V)\to L^2(\mathbb{R}^d;\mu_V)}:=&\sup_{f\in L^\infty(\mathbb{R}^d;\mu_V)}\|P_tf-\mu_V(f)\|_{L^2(\mathbb{R}^d;\mu_V)}\\
\leqslant&\ \lambda t^{-\varepsilon_*/\sum_{i=1}^d(\alpha-\varepsilon_i)^+},\quad t>0.
\end{aligned}$$

In particular, let $e^{-V(x)}$ *be the density function above with* $\varepsilon_k\in(0,\alpha)$ *for some* $1\leqslant k\leqslant d$ *and* $\varepsilon_i\in[\alpha,\infty)$ *for any* $i\neq k$. *Then, the rate function* α *given by* (3.21) *is*

$$\eta(r)=c_3\Big(1+r^{-(\alpha-\varepsilon_k)/\varepsilon_k}\Big),\quad r>0$$

for some constant $c_3>0$, *which is sharp in the sense that* (3.20) *does not hold if*

$$\lim_{r\to0}r^{(\alpha-\varepsilon_k)/\varepsilon_k}\eta(r)=0.$$

(2) *Let*

$$e^{-V(x)}=C_{\varepsilon_1,\ldots,\varepsilon_d,\alpha}\prod_{i=1}^d(1+|x_i|)^{-(1+\alpha)}\log^{-\varepsilon_i}(e+|x_i|)$$

with $\varepsilon_*:=\min_{1\leqslant i\leqslant d}\varepsilon_i<0$. *Then the weak Poincaré inequality* (3.20) *holds with*

$$\eta(r)=c\Big(1+\log^{-\sum_{i=1}^d(\varepsilon_i\wedge0)}(1+r^{-1})\Big),\quad r>0 \tag{3.22}$$

for some constant $c>0$. Consequently, there exist constants λ_1 and $\lambda_2>0$ such that

$$\|P_t-\mu_V\|_{L^\infty(\mathbb{R}^d;\mu_V)\to L^2(\mathbb{R}^d;\mu_V)}:=\sup_{f\in L^\infty(\mathbb{R}^d;\mu_V)}\|P_tf-\mu_V(f)\|_{L^2(\mathbb{R}^d;\mu_V)}$$
$$\leqslant \exp\left(\lambda_1-\lambda_2 t^{1/(1-\sum_{i=1}^d(\varepsilon_i\wedge 0))}\right),\quad t>0.$$

In particular, consider the density function

$$e^{-V(x)}=C\left(\prod_{i=1}^d(1+|x_i|)^{-(1+\alpha)}\right)\left(\log^{\theta_k}(e+|x_k|)\prod_{1\leqslant i\leqslant d,i\neq k}\log^{-\varepsilon_i}(e+|x_i|)\right)$$

on $\mathbb{R}^d$ with $\theta_k>0$, $\varepsilon_i\geqslant 0$ for some $1\leqslant k\leqslant d$ and any $i\neq k$ and C is the normalizing constant. Then, the rate function α defined by (3.22) *is reduced into*

$$\eta(r)=c\Big(1+\log^{\theta_k}(1+r^{-1})\Big),\quad r>0$$

with some constant $c>0$, which is sharp in the sense that (3.20) *does not hold if*

$$\lim_{r\to 0}\log^{-\theta_k}(1+r^{-1})\eta(r)=0.$$

Proof. (1) Let

$$e^{-V_0(x)}=C_{\varepsilon_1,\ldots,\varepsilon_d,\alpha}\prod_{i=1}^d(1+|x_i|)^{-(1+(\varepsilon_i\vee\alpha))}.$$

Then, (3.19) holds. On the other hand, for any $s>0$, let

$$A_s=\{x\in\mathbb{R}^d:|x_i|\leqslant s\text{ for }1\leqslant i\leqslant d\text{ such that }\varepsilon_i<\alpha\}.$$

Then, there is a constant $c_1>0$ such that for all $s>0$,

$$\Phi_0(s)=c_1(1+s)^{\sum_{i=1}^d(\alpha-\varepsilon_i)^+}.$$

Hence, for $r>0$ small enough,

$$\eta(r)\leqslant c_2\inf\left\{(1+s)^{\sum_{i=1}^d(\alpha-\varepsilon_i)^+}:s>0\text{ and }\sum_{i:0<\varepsilon_i<\alpha}\frac{1}{(1+s)^{\varepsilon_i}}\leqslant c_3r\right\}.$$

Setting $s=c_4r^{-1/\varepsilon*}$ for $r>0$ small enough and some constant $c_4>0$ in the right hand side, we can get the first assertion by Theorem 3.3. Furthermore, the corresponding bound of $\|P_t-\mu_V\|_{L^\infty(\mathbb{R}^d;\mu_V)\to L^2(\mathbb{R}^d;\mu_V)}$ follows from [16, Theorem 4.1.5(2)]. Here we mention that, since $\|P_tf-\mu_Vf\|_{L^2(\mathbb{R}^d;\mu_V)}\leqslant 2\|f\|_{L^\infty(\mathbb{R}^d;\mu_V)}$ for all $t>0$, the bound in t is useful only for large t. Suppose that the weak Poincaré inequality (3.20) holds for a probability measure

$$e^{-V(x)}=C_{\varepsilon_1,\ldots,\varepsilon_d}(1+|x_k|)^{-(1+\varepsilon_k)}\prod_{1\leqslant i\leqslant d,i\neq k}(1+|x_i|)^{-(1+\varepsilon_i)}$$

on $\mathbb{R}^d$ with $\varepsilon_k\in(0,\alpha)$ and $\varepsilon_i\in[\alpha,\infty)$ for some $1\leqslant k\leqslant d$ and any $i\neq k$. We consider the function $f(x)=g(x_k)$, where $g\in C_c^\infty(\mathbb{R})$. Also by [5, Theorem 2.1(1)], we can apply $f\in C_b^\infty(\mathbb{R}^d)\subset\mathscr{D}(D)$ into the weak Poincaré inequality (3.20), and

obtain that for this class of functions the weak Poincaré inequality (3.20) is reduced into

$$
\begin{aligned}
m(g^2) \leqslant & \eta(r) \int_{\mathbb{R}\times\mathbb{R}} \frac{(g(x+z)-g(x))^2}{|z|^{1+\alpha}}\, dz\, m(dx) \\
& + r\|g\|_\infty^2, \quad m(g)=0, g\in C_c^\infty(\mathbb{R}),
\end{aligned} \tag{3.23}
$$

where $m(dx) = C_{\varepsilon_k}(1+|x|)^{-(1+\varepsilon_k)}\, dx$ and $\varepsilon_k \in (0,\alpha)$. Then, the last assertion is a consequence of [17, Corollary 1.2(3)].

(2) Let

$$
e^{-V_0(x)} = C_{\varepsilon_1,\dots,\varepsilon_d,\alpha} \prod_{i=1}^d (1+|x_i|)^{-(1+\alpha)} \log^{-(\varepsilon_i\vee 0)}(e+|x_i|).
$$

Then, (3.19) holds. On the other hand, for any $s>0$, let

$$
A_s = \{x\in\mathbb{R}^d : |x_i| \leqslant s \text{ for } 1\leqslant i\leqslant d \text{ such that } \varepsilon_i<0\}.
$$

We can find some constant $c_1>0$ such that for all $s>0$,

$$
\Phi_0(s) = c_1\Big(\log(e+s)\Big)^{-\sum_{i=1}^d(\varepsilon_i\wedge 0)},
$$

where $-\sum_{i=1}^d(\varepsilon_i\wedge 0)>0$. Then, for $r>0$ small enough,

$$
\eta(r) \leqslant c_2 \inf\left\{\Big(\log(e+s)\Big)^{-\sum_{i=1}^d(\varepsilon_i\wedge 0)} : s>0 \text{ and } \sum_{i:\varepsilon_i<0} \frac{1}{(1+s)^\alpha \log^{\varepsilon_i}(e+s)} \leqslant c_3 r\right\}.
$$

Therefore, we can prove the first assertion, by using Theorem 3.3 and taking $s = c_4\Big(\frac{1}{r}\log^{-\varepsilon_*}(1+\frac{1}{r})\Big)^{1/\alpha}$ for $r>0$ small enough and some proper constant $c_4>0$ in the right hand side of the inequality above.

Furthermore, the corresponding bound of $\|P_t-\mu_V\|_{L^\infty(\mathbb{R}^d;\mu_V)\to L^2(\mathbb{R}^d;\mu_V)}$ follows from [16, Theorem 4.1.5(1)]. Suppose that the weak Poincaré inequality (3.20) holds for a probability measure with the density function as follows

$$
e^{-V(x)} = C\left(\prod_{i=1}^d (1+|x_i|)^{-(1+\alpha)}\right)\left(\log^{\theta_k}(e+|x_k|)\prod_{1\leqslant i\leqslant d, i\neq k}\log^{-\varepsilon_i}(e+|x_i|)\right),
$$

where $\theta_k>0$ and $\varepsilon_i\geqslant 0$ for some $1\leqslant k\leqslant d$ and any $i\neq k$. We consider the function $f(x)=g(x_k)$, where $g\in C_c^\infty(\mathbb{R})$. Then, the weak Poincaré inequality (3.20) is reduced into the inequality (3.23), where

$$
m(dx) = C_{\theta_k,\alpha}(1+|x|)^{-(1+\alpha)}\log^{\theta_k}(e+|x|)\, dx
$$

and $\theta_k>0$. Hence, the last assertion follows from [17, Corollary 1.3(4)]. $\square$

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